

Antisymmetric tensor matter fields in a curved space-time

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Abstract

An analysis about the antisymmetric tensor matter fields Avdeev-Chizhov theory in a curved space-time is performed. We show, in a curved spacetime, that the Avdeev-Chizhov theory can be seen as a kind of a $\lambda\varphi^4$ theory for a "complex self-dual" field. This relationship between Avdeev-Chizhov theory and $\lambda\varphi^4$ theory simplify the study of tensor matter fields in a curved space-time. The energy-momentum tensor for matter fields is computed.

1 Introduction

Antisymmetric tensor fields have been introduced since many years and are object of continuous and renewed interests due to their connection with the topological field theories [1, 2, 3, 4]. In 1994 L.V. Avdeev and M.V. Chizhov [5] achieved the construction of a four dimensional abelian gauge model which includes antisymmetric second rank tensor fields as matter fields rather than gauge fields. The model contains also a coupling of the antisymmetric fields with chiral spinors and a quartic tensor self-interaction term. It exhibits several features among which we underline the asymptotically free ultraviolet behavior of the abelian gauge interaction. More recently V. Lemes, R.R. Landim and S.P. Sorella [6] found the interesting result that the tensor matter invariant lagrangian proposed for Avdeev-Chizhov can be seen as a kind of a $\lambda\varphi^4$ theory for a complex antisymmetric field satisfying a "complex self-dual condition" in Minkowski spacetime, give us a straightforward way of obtaining its nonabelian generalization.

In this work we shall to generalize the Avdeev-Chizhov theory for curved space-times. It would be a very tiresome task. Therefore we shall begin this analysis studying the relationship between Avdeev-Chizhov theory and $\lambda\varphi^4$ theory in a curved space-time. We shall show that the equivalence between Avdeev-Chizhov theory and $\lambda\varphi^4$ theory archived for [6] in a flat space-time remains if a curvature is introduced. As we shall see, It will simplify the study of tensor matter fields in a curved space-time.

2 Complex self-dual condition in a curved space-time

The self-dual complex condition introduced in [6] depend on signature and dimension of space-time. In this section we shall investigate the self-dual complex condition in a four dimensional curved space-time with signature -2, which is used in general relativity.

The dual of a tensor $F_{\mu\nu}$ is defined as:

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}, \quad (1)$$

with

$$\epsilon_{\mu\nu\rho\sigma} = \sqrt{-g}\varepsilon_{\mu\nu\rho\sigma} \quad ; \quad \epsilon^{\mu\nu\rho\sigma} = \frac{1}{\sqrt{-g}}\varepsilon^{\mu\nu\rho\sigma}, \quad (2)$$

where $\varepsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita tensor.

Let us build the complex field $\varphi_{\mu\nu}$ like in the flat case [6]:

$$\varphi_{\mu\nu} = T_{\mu\nu} + i\tilde{T}_{\mu\nu}. \quad (3)$$

Then, we have:

$$\tilde{\varphi}_{\mu\nu} = \tilde{T}_{\mu\nu} + i\tilde{\tilde{T}}_{\mu\nu},$$

where $\tilde{\tilde{T}}_{\mu\nu}$ is given for (1) and

$$\tilde{\tilde{T}}_{\mu\nu} = -T_{\mu\nu}. \quad (4)$$

Hence, we have

$$\tilde{\varphi}_{\mu\nu} = \tilde{T}_{\mu\nu} - iT_{\mu\nu} \rightarrow i\tilde{\varphi}_{\mu\nu} = T_{\mu\nu} + i\tilde{T}_{\mu\nu} = \varphi_{\mu\nu}$$

or

$$i\tilde{\varphi}_{\mu\nu} = \varphi_{\mu\nu}. \quad (5)$$

Then we can build complex self-dual fields in a four dimensional curved space-times with signature -2.

3 Tensor matter fields as a $\lambda\varphi^4$ theory in a curved space-time

We shall show now that the Avdeev-Chizhov theory for matter fields can be seen as a kind of $\lambda\varphi^4$ theory in a curved space-time for a self-dual complex field $\varphi_{\mu\nu}$.

Firstly, let us write the action for complex field $\varphi_{\mu\nu}$ in Minkowski space-time:

$$\begin{aligned} S_{inv} &= \int d^4x - \frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + i\bar{\psi}\gamma^\mu \partial_\mu \psi - \bar{\psi}\gamma^\mu \gamma_5 A_\mu \psi \\ &- \int d^4x \left((\varphi_{|\mu}^{\mu\nu})^* (\varphi_{\sigma\nu}^{|\sigma}) + \frac{q}{8} (\varphi^{*\mu\nu} \varphi_{\nu\alpha} \varphi^{*\alpha\beta} \varphi_{\beta\mu}) \right) \\ &+ \int d^4x \frac{1}{2} y \bar{\psi} \sigma_{\mu\nu} (\varphi^{*\mu\nu} + \varphi^{\mu\nu}) \psi, \end{aligned} \quad (6)$$

where $\varphi_{|}$ indicates the gauge covariante derivative of φ field [7], which we go to define as

$$\varphi_{\mu\nu | \alpha} = -i(\varphi_{\mu\nu} , \alpha - 2iA_\alpha \varphi_{\mu\nu}), \quad (7)$$

what is justified by the fact $\varphi_{|}$ transforms under gauge transformations (12) as the fields φ .

The action (7) is invariant under the infinitesimal abelian gauge transformations:

$$\begin{aligned} \delta A_\mu &= \partial_\mu \omega, & \delta \psi &= -i\omega \gamma_5 \psi, \\ \delta \bar{\psi} &= -i\omega \bar{\psi} \gamma_5, & \delta \varphi_{\mu\nu} &= -2\omega \varphi_{\mu\nu}. \end{aligned} \quad (8)$$

Let us now, using the general covariance principle, perform the following transformations:

$$\begin{aligned} \eta_{\mu\nu} &\rightarrow g_{\mu\nu} \quad ; \quad d^4x \rightarrow \sqrt{-g} d^4x \\ \varphi^{\mu\nu}, \alpha &\rightarrow \varphi^{\mu\nu}; \alpha \quad ; \quad \varphi^{\mu\nu}_{|\alpha} \rightarrow \varphi^{\mu\nu}_{||\alpha} \\ \gamma^\mu &\rightarrow \rho^\mu \quad ; \quad \psi, \mu \rightarrow \psi; \mu, \end{aligned}$$

where $\varphi_{||}$ indicates the gauge and coordinate covariante derivative of φ field [7], which is defined as

$$\varphi_{\mu\nu || \alpha} = -i(\varphi_{\mu\nu}; \alpha - 2iA_\alpha \varphi_{\mu\nu}) \quad (9)$$

Let us also include the gravitational action

$$S_g = \int \sqrt{-g} R d^4x$$

where $R = g_{\mu\nu} R^{\mu\nu}$ is the Ricci scalar.

Then, the action for the $\varphi_{\mu\nu}$ field in a curved space-time is :

$$\begin{aligned} S_{inv} &= \int \sqrt{-g} d^4x \left(\frac{1}{16\pi G} R - \frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + i\rho^\mu \psi; \mu - \bar{\psi} \rho^\mu \gamma_5 A_\mu \psi - (\varphi_{||\mu}^{\mu\nu})^* (\varphi_{\sigma\nu}^{||\sigma}) \right. \\ &\quad \left. + \frac{q}{8} (\varphi^{*\mu\nu} \varphi_{\nu\alpha} \varphi^{*\alpha\beta} \varphi_{\beta\mu}) + \frac{1}{2} y \bar{\psi} \sigma_{\mu\nu} (\varphi^{*\mu\nu} + \varphi^{\mu\nu}) \psi \right). \end{aligned} \quad (10)$$

The action (10) is, like in the flat case, invariant under the infinitesimal abelian gauge transformations: :

$$\begin{aligned} \delta A_\mu &= \omega_{, \mu} = \omega_{, \mu} \quad ; \quad \delta \psi = -i\omega \gamma_5 \psi \\ \delta \bar{\psi} &= -i\omega \bar{\psi} \gamma_5 \quad ; \quad \delta \varphi_{\mu\nu} = -2\omega \varphi_{\mu\nu}. \end{aligned} \quad (11)$$

Writing $\varphi_{\mu\nu}$ as

$$\varphi_{\mu\nu} = T_{\mu\nu} + i\tilde{T}_{\mu\nu}, \quad (12)$$

and using the following identities :

$$\tilde{T}_{\mu\lambda} \tilde{T}^{\lambda\nu} = T_{\mu\lambda} T^{\lambda\nu} + \frac{1}{2} \delta_\mu^\nu T_{\alpha\beta} T^{\alpha\beta},$$

$$T_{\mu\lambda} \tilde{T}^{\lambda\nu} = -\frac{1}{4} \delta_\mu^\nu T_{\alpha\beta} \tilde{T}^{\beta\alpha}, \quad (13)$$

$$\tilde{T}_{\mu\rho} \tilde{T}^{\sigma\rho}_{; \sigma} = -\frac{1}{2} T^{\alpha\beta} T_{\alpha\beta; \mu} - (T_{\lambda\mu; \sigma}) T^{\sigma\lambda}, \quad (14)$$

$$\tilde{T}_{\alpha\beta} T^{\alpha\beta; \mu}_{; \mu} = -2T_{\alpha\beta} \tilde{T}^{\nu\alpha; \beta}_{; \nu} - 2\tilde{T}_{\alpha\beta} T^{\nu\alpha; \beta}_{; \nu}, \quad (15)$$

after a straightforward calculation, we thus arrive at the following action

$$\begin{aligned} S_{inv} = \int \sqrt{-g} d^4x (& \frac{1}{16\pi G} R - \frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + i\rho^\mu \psi_{; \mu} - \rho^\mu \gamma_5 A_\mu \psi + \frac{1}{2} (T_{\mu\nu; \lambda})^2 - 2(T_{\mu\nu; \mu})^2 \\ & + 4A_\mu (T^{\mu\nu} \tilde{T}_{\lambda\nu}^{; \lambda} - \tilde{T}^{\mu\nu} T_{\lambda\nu}^{; \lambda}) + 4(\frac{1}{2} (A_\lambda T_{\mu\nu})^2 - 2(A^\mu T_{\mu\nu})^2) \\ & + y \bar{\psi} \sigma_{\mu\nu} T^{\mu\nu} \psi + \frac{q}{4} (\frac{1}{2} (T_{\mu\nu} T^{\mu\nu})^2 - 2T_{\mu\nu} T^{\nu\rho} T_{\rho\lambda} T^{\lambda\mu})), \end{aligned}$$

and

$$\begin{aligned} \delta A_\mu &= \omega_{, \mu} = \omega_{, \mu} \quad ; \quad \delta \psi = -i\omega \gamma_5 \psi \\ \delta \bar{\psi} &= -i\omega \bar{\psi} \gamma_5 \quad ; \quad \delta T_{\mu\nu} = -2\omega \tilde{T}_{\mu\nu}. \end{aligned} \quad (16)$$

The action above is a Avdeev-Chizhov like action in a curved space-time.

4 Energy-momentum tensor

In this section we shall to compute the energy-momentum tensor for antisymmetric matter fields. As we shall see, the relationship between Avdeev-Chizhov and $\lambda\varphi^4$ theory simplify this.

The energy-momentum tensor is very important in general relativity for describe the matter distribution which acts as sources for the gravitational field. It is defined as [7]

$$\Theta_{\mu\nu} = \frac{2}{\sqrt{-g}} \left\{ \frac{\partial(\sqrt{-g}L)}{\partial g^{\mu\nu}} - \frac{\partial}{\partial x^\alpha} \left[\frac{\partial(\sqrt{-g}L)}{\partial g^{\mu\nu}_{, \alpha}} \right] \right\}. \quad (17)$$

The direct use of Avdeev-Chizhov Lagrangian in (17) would lead to very tiresome computations. Then let us use the Lagrangian for $\varphi_{\mu\nu}$ rather the Lagrangian for $T_{\mu\nu}$ field.

Firstly, let us analyze the second term of right side of (17), which is what it gives more work to calculate. We have that only the kinetic term of (10) could depend on derivatives of the metric in relation to the coordinates. However, using the self-dual complex condition (5) and the following identity:

$$\widehat{\varphi}^{\alpha\beta}_{||\alpha} = \frac{1}{2}\epsilon^{\alpha\beta\mu\nu}\varphi_{\mu\nu| \alpha}, \quad (18)$$

we have that:

$$\begin{aligned} (\varphi^{\mu\nu}_{||\mu})^*(\varphi_{\sigma\nu}^{||\sigma}) &= g^{\sigma\omega}g^{\rho\phi}g^{\tau\gamma}\epsilon^{\mu\nu\alpha\beta}\epsilon_{\omega\nu\phi\gamma}(\varphi_{\alpha\beta| \mu})^*(\varphi_{\rho\tau| \sigma}) \\ &= \frac{-1}{4}g^{\sigma\omega}g^{\rho\phi}g^{\tau\gamma}\delta_{\omega\phi\gamma}^{\mu\alpha\beta}(\varphi_{\alpha\beta| \mu})^*(\varphi_{\rho\tau| \sigma}). \end{aligned}$$

As we see, the Lagrangian for $\varphi^{\mu\nu}$ field don't depend on metric derivatives. Hence the second term of right side of (17) vanish.

Computing the other term we found for energy-momentum tensor:

$$\Theta_{\mu\nu} = 3g^{\rho\phi}(\varphi_{[\phi\nu, \omega]})^*(\varphi_{\rho\mu}^{, \omega} - \frac{1}{2}g^{\tau\omega}\varphi_{\rho\tau| \mu}) \quad (19)$$

$$\begin{aligned} &- \frac{q}{8}(\varphi^{*\lambda\gamma}\varphi_{\gamma\mu}\varphi_{\nu}^{*\beta}\varphi_{\beta\lambda} + 3\varphi^{*\lambda\gamma}\varphi_{\gamma}^{\sigma}\varphi_{\sigma\nu}^*\varphi_{\mu\lambda}) \\ &- \frac{1}{2}g_{\mu\nu}L + (\mu \leftrightarrow \nu). \end{aligned} \quad (20)$$

In terms of matter field $T_{\mu\nu}$ we have:

$$\begin{aligned} \Theta_{\mu\nu} = & - (3/2)g^{\rho\phi}T_{[\omega\phi, \nu]}(T_{\rho\mu}^{, \omega} - \frac{1}{2}g^{\tau\omega}T_{\rho\tau| \mu}) + (T \rightarrow \widetilde{T}) \\ & + \frac{q}{2}(T_{\mu}^{\lambda}T_{\nu\lambda}T^{\omega\phi}T_{\omega\phi} + 4T_{\beta\nu}T_{\mu\lambda}T^{\lambda\alpha}T_{\alpha}^{\beta}) \\ & - \frac{1}{2}g_{\mu\nu}L + (\mu \leftrightarrow \nu). \end{aligned}$$

If we include the interactions of tensor matter fields with eletromagnetic field, we shall to find

$$\Theta_{\mu\nu} = 3g^{\rho\phi}(\varphi_{[\phi\nu| \omega]})^*(\varphi_{\rho\mu}^{, \omega} - \frac{1}{2}g^{\tau\omega}\varphi_{\rho\tau| \mu}) \quad (21)$$

$$\begin{aligned} &- \frac{q}{8}(\varphi^{*\lambda\gamma}\varphi_{\gamma\mu}\varphi_{\nu}^{*\beta}\varphi_{\beta\lambda} + 3\varphi^{*\lambda\gamma}\varphi_{\gamma}^{\sigma}\varphi_{\sigma\nu}^*\varphi_{\mu\lambda}) \\ &+ \frac{1}{4h^2}\sqrt{-g}\left(\frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - F_{\mu\alpha}F_{\nu}^{\alpha}\right) - \frac{1}{2}g_{\mu\nu}L + (\mu \leftrightarrow \nu). \end{aligned} \quad (22)$$

It's easy to verify that $\Theta_{\mu\nu}$ is invariant under gauge transformations (12)

$$\delta\Theta_{\mu\nu} = 0. \quad (23)$$

5 Conclusion

We have shown here that Avdeev-Chizhov theory for antisymmetric tensor matter fields can be seen as a kind of $\lambda\varphi^4$ theory for a self-dual complex field in a curved space-time. Also we have shown that the relationship above simplifies the computation of energy-momentum tensor, fundamental object in general relativity, for matter fields in a curved space-time, making the study of Avdeev-Chizhov theory in a curved space-time easier.

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References

- [1] D. Birmingham, M. Blau, M. Rakowski, and G. Thompson. Topological field-theory. *Phys. Rep.-Rev. Sec. Phys. Lett.*, 209(4-5):129–340, 1991.
- [2] G.T Horowitz. Exatily soluble diffeomorphism invariant theories. *Comm. Math. Phys.*, 125(3):417–437, 1989.
- [3] G.T. Horowitz and M. Srednicki. A quantum field theoretic description of linking numbers and their generalization. *Comm. Math. Phys.*, 130(1):83–94, 1990.
- [4] M. Blau and G. Thompson. Topological gauge-theories of antisymmetric tensor-fields. *Ann. Phys.*, 205(1):130–172, 1991.
- [5] L. V. Avdeev and M. V. Chizhov. Antisymmetric tensor matter fields - an abelian model. *Phys. Lett. B*, 321(3):212–218, 1994.
- [6] V. Lemes, R. Renan, and S. P. Sorella. ϕ_4^4 -theory for antisymmetric tensor matter fields in Minkowski space-time. *Phys. Lett. B*, 352(1-2):37–42, 1995.
- [7] M. Carmeli. *Classical Fields: General Relativity and Gauge Theory*. John Wiley and Sons, Inc, 1982.